UNCERTAINTY OF A STATE VERTICAL REFERENCE SYSTEM BELOW 1 MGPU – IS IT POSSIBLE?

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Since the end of the 18th century, the basic method for establishing state or continental vertical reference systems has been the precise geometric levelling. Since then, it is supposed that the accuracy of this method is a function of the levelling distance [1, 2, 3]. As a result, the weights applied in the levelling network adjustment are some functions of the levelling line lengths. In addition, based on the Gauss law of error propagation, the mean of both elevation measurements in levelling lines has been preferred as a more plausible value of the measured line elevation.

However, the modern probability theory [4] and popular statistical methods [5] do not support the above levelling assumptions. According to [4], the probability *Cv(n)* that the average of *n* independent random variables is closer to the distribution expectation than some of the independent variables is a function of the entropy of the standard Normal distribution and the number of variables *n*. The probability *Cv(n)* can be given by equation (1), where the coefficients *a, b,* and *c* depend on the distribution parameters and the number of variables *n*.

$$Cv(n) = 0.25.log_b(2\pi e).(n^{-c} - n^{-a}) \quad (1)$$

Based on (1), it can be calculated that in the case of two measurements, i.e., $n = 2$, the probability *Cv(2)* tends to 30% if both measurements are normally distributed. Suppose the situation where both measurements derive from the distribution with the highest entropy, i.e., the Uniform distribution, then *Cv(2)* tends to 33%. Thus, in more than 66% of the cases, the value of one of the two measurements is closer to the true value of the measured quantity in comparison to the mean. Figure 1 illustrates the frequencies of occurrence of the first, the second observation, or their mean, most closely to a known expectation for various distributions [6].

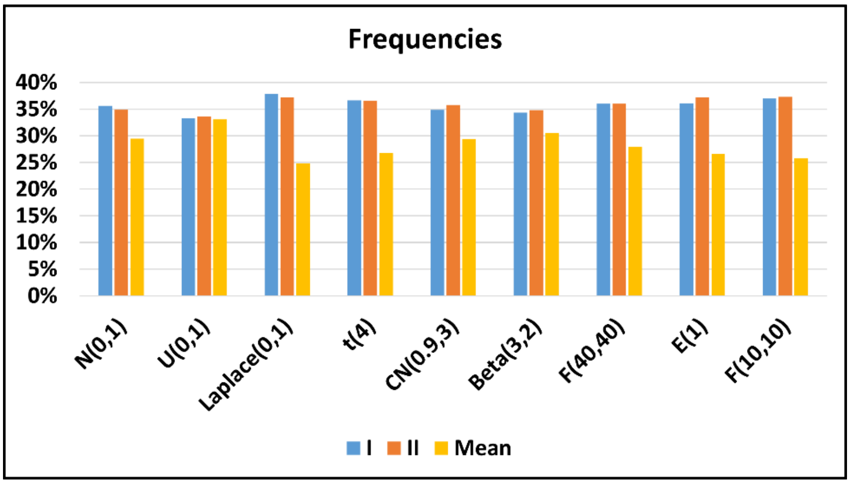

Fig. 1. The frequencies of occurrence of the first, the second observation, or their mean most closely to a known expectation of the applied distributions [6]

The regression analysis of the data obtained in the Third Levelling of Bulgaria /1975-1984/, the Second Levelling of Finland /1935-1955/ [2] and the Third Levelling of Finland /1978-2006/ performed in the study [5] shows that the adjusted coefficient of determination of the absolute discrepancies in levelling lines in respect to the square root of their length $\sqrt{L}$, their length L, the sum of the absolute elevations along the lines $\sum|h|$, and the absolute elevation between line terminal benchmarks |H| are 0.28, 0.34, and 0.26, respectively. In each of the cases, the most significant factor for forming |D|, considering equation (2) is the sum of the absolute elevation values along the lines $\sum|h|$, which presents the terrain complexity along the levelling lines.

$$|D| = a + b.\sqrt{L} + c.L + d.\sum|h| + e.|H| \tag{11}$$

According to the study [7], there is no significant correlation between the closing errors $|W|$, the loop circumferences L and the square root of L in the above-mentioned levelling networks.

Table 1. Correlation coefficients among the absolute values of the closing errors $|W|$, loop circumferences L and the square root of L in the analysed precise levelling networks [7]

Network	$\boldsymbol{\rho_{|W|,L}}$	$\boldsymbol{\rho_{|W|,\sqrt{L}}}$	$\boldsymbol{\rho_{L,\sqrt{L}}}$
	unitless	unitless	unitless
The Third Levelling of Bulgaria /1975-1984/	0.349	0.348	0.990
The Second Levelling of Finland /1935-1955/	0.104	0.095	0.997
The Third Levelling of Finland /1978-2006/	-0.006	-0.006	0.993

Taking into account all revealed facts, we adjusted the part of the Third Levelling of Finland [3, 6], presented in Figure 2, using a new approach. The basic steps are as follows:

1. Using 3^n independent adjustments, in our case 3^{18} = 387,420,489 adjustments, we selected those values of line elevations among the forwards, the backwards, and their means, which minimised the loop closing errors [6].
2. Using the selected line elevations, we performed an additional 20 independent adjustments to estimate the impact of each line elevation on the network accuracy. In each adjustment, we skipped a different line, and we calculated the sum of the standard errors of the adjusted benchmark geopotential numbers. We skipped a different line in different adjustments and assessed the network accuracy. Based on these results, we formed our weights for each line in the network as a function of the produced accuracy. If a skipped line leads to higher accuracy, we gave it a greater weight in the final adjustment. The weights of each line were calculated as the square of the ratio between the average of the benchmark standard errors in each variant and the average of the benchmark standard errors in all variants.
3. Finally, using the selected elevations in step 1 and their non-parametric and assumption-free weights, obtained as described in step 2, we adjusted the network in Figure 2 as a free levelling network /without a datum point/. Our decision was provoked by the wish to refer the standard errors of the adjusted benchmark geopotential numbers to the network weight centre.

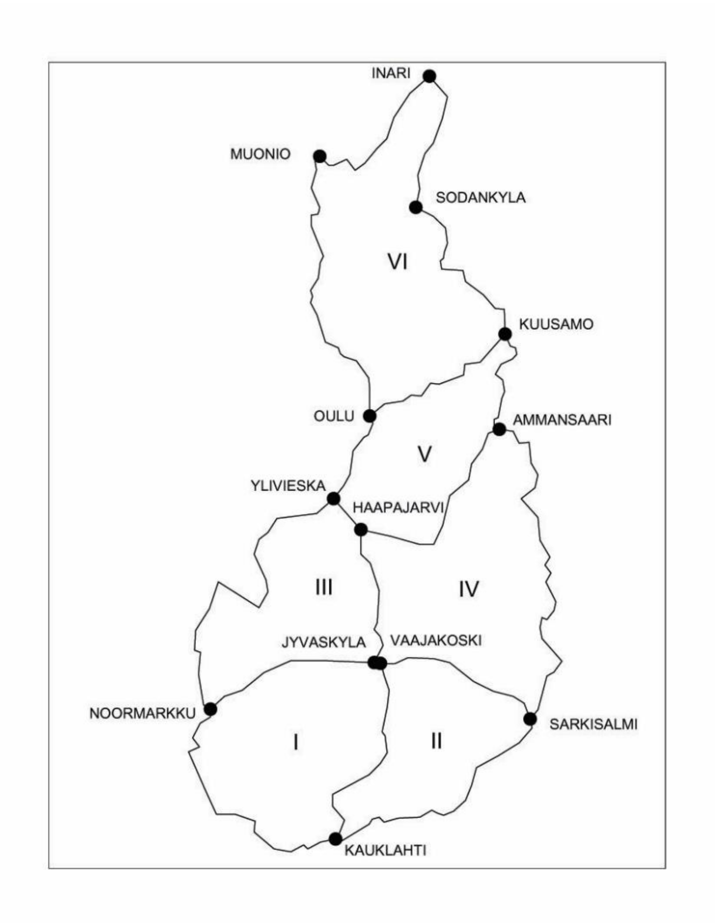

Fig. 2. Scheme of the analysed network, part of the Third Level of Finland network [3, 6]

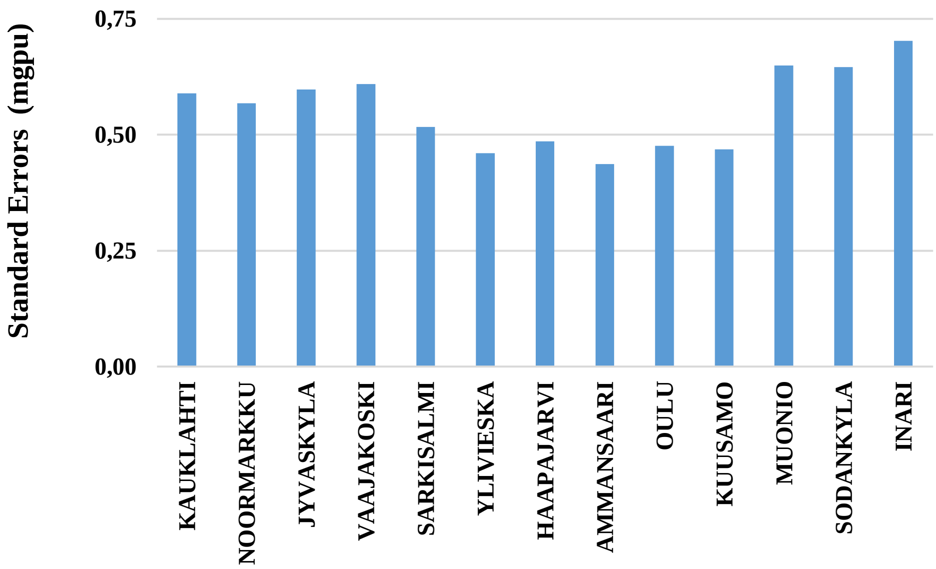

Fig. 3. Standard Errors of the adjusted benchmark geopotential numbers in mgpu

As can be seen from Figure 3, the standard errors of all adjusted benchmark geopotential numbers are below 0.75 mgpu. The adjusted geopotential number of the Ammansaari benchmark has a minimal standard error of 0.44 mgpu. This benchmark is close to the network weight centre. The benchmarks located in the periphery of the network, e.g., Inari, Sodankyla, and Mounio have greater standard errors of the adjusted geopotential numbers, respectively 0.70, 0.65, and 0.65 mgpu. The mean standard error is 0.55 mgpu. The standard deviation of the standard error sample is 0.09 mgpu. Comparison between the standard errors of the adjusted geopotential numbers pictured in Fig. 3 and the standard errors of the adjusted geopotential number of the same benchmarks, but interpolated by Figure 6.3 in the study [3] shows that the uncertainty of a levelling network, adjusted in the manner presented here, can be reduced more than 15-20 times. Taking this fact into account, plus the current progress of the GNSS technologies [1, 8, 9, 10], it is likely that geoid-based vertical reference frames with uncertainty below 10 mgpu can be realised soon.

Acknowledgements

The authors express gratitude to the Centre for Scientific Research and Design at the UACEG for funding and support for the completed research project under contract No BN315/2025 on the topic "Optimisation of levelling networks by studying the applicability of resampling methods."

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